A KRONECKER-WEYL THEOREM FOR SUBSETS OF ABELIAN GROUPS

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ABSTRACT. Let $\mathbb{N}$ be the set of non-negative integer numbers, $\mathbb{T}$ the circle group and $\mathfrak{c}$ the cardinality of the continuum. Given an abelian group G of size at most $2^{\mathfrak{c}}$ and a countable family $\mathscr{E}$ of infinite subsets of G, we construct "Baire many" monomorphisms $\pi : G \to \mathbb{T}^{\mathfrak{c}}$ such that $\pi(E)$ is dense in $\{y \in \mathbb{T}^{\mathfrak{c}} : ny = 0\}$ whenever $n \in \mathbb{N}$, $E \in \mathscr{E}$, $nE = \{0\}$ and $\{x \in E : mx = g\}$ is finite for all $g \in G$ and m such that $n = mk$ for some $k \in \mathbb{N} \setminus \{1\}$. We apply this result to obtain an algebraic description of countable potentially dense subsets of abelian groups, thereby making a significant progress towards a solution of a problem of Markov going back to 1944. A particular case of our result yields a positive answer to a problem of Tkachenko and Yaschenko [23, Problem 6.5]. Applications to group actions and discrete flows on $\mathbb{T}^{\mathfrak{c}}$, diophantine approximation, Bohr topologies and Bohr compactifications are also provided.

We refer the reader to [17] for a background on abelian groups. All undefined topological terms can be found in [15].

We use $\mathbb{N}$ and $\mathbb{N}^{+}$ to denote the set of all natural numbers and positive natural numbers, respectively. The groups of integer numbers and real numbers are denoted by $\mathbb{Z}$ and $\mathbb{R}$, respectively. We use $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ to denote the circle group (written additively). As usual, the symbol $|X|$ stands for the cardinality of a set X, and we let $\omega = |\mathbb{N}|$ and $\mathfrak{c} = |\mathbb{R}|$.

Let G be an abelian group. For every $m \in \mathbb{N}$, define $mG = \{mx : x \in G\}$ and $G[m] = \{x \in G : mx = 0\}$. We say that G is *bounded* if $G = G[n]$ for some $n \in \mathbb{N}^{+}$, and the minimal such n is called the *exponent* of G. If $nG = G$ for every $n \in \mathbb{N}^{+}$, then G is said to be *divisible*. We denote by $r_0(G)$ the free rank of the group G and by $r_p(G)$ the p-rank of G for a prime number p. For a compact Hausdorff abelian group K, we use $\mathrm{Hom}(G, K)$ to denote the set of all group homomorphisms from G to K equipped with the topology of pointwise convergence, i.e., with the subspace topology that $\mathrm{Hom}(G, K)$ inherits from the Tychonoff product topology on K^{G}. Since $\mathrm{Hom}(G, K)$ is closed in the compact space K^{G}, $\mathrm{Hom}(G, K)$ is a compact Hausdorff (abelian) group, with pointwise addition of homomorphisms as the group operation. The set of all monomorphisms from G to K is denoted by $\mathrm{Mono}(G, K)$. Recall that $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ is the Pontryagin dual of (the discrete abelian group) G.

A topological group is called *precompact* (or *totally bounded*) if its completion is compact [24].

1. INTRODUCTION

Let $\mathbb{C}$ denote the complex plane and $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. For an abelian group K, a map $\chi : K \to \mathbb{S}$ will be called an $\mathbb{S}$-*character* of K provided that $\chi(0) = 1$ and $\chi(x + y) = \chi(x) \cdot \chi(y)$ whenever $x, y \in K$.

Let K be a compact abelian group and let μ be its Haar measure. A one-to-one sequence $\{x_n : n \in \mathbb{N}\}$ in K is called *uniformly distributed* provided that

$$(1) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f(x_j) = \int_K f \, d\mu \quad \text{for every continuous function } f : K \to \mathbb{C}.$$

If $\chi : K \to \mathbb{C}$ is a non-trivial continuous $\mathbb{S}$-character, then $\int_K \chi \, d\mu = 0$, and so the following criterion due to Weyl says that it suffices to take as f in (1) only non-trivial continuous $\mathbb{S}$-characters: A one-to-one sequence $\{x_n : n \in \mathbb{N}\}$ in K is uniformly distributed if and only if $\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \chi(x_j) = 0$ for every non-trivial continuous $\mathbb{S}$-character χ of K. Since the only continuous $\mathbb{S}$-characters of $\mathbb{T}$ are of the form $x \mapsto e^{2\pi i k x}$ for some $k \in \mathbb{Z}$, for $K = \mathbb{T}$ the Weyl criterion becomes: A sequence $\{x_n : n \in \mathbb{N}\}$ in $\mathbb{T}$ is uniformly distributed if and only if $\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i k x_j} = 0$ for every $k \in \mathbb{Z}$.

For a strictly increasing sequence $S = \{a_n : n \in \mathbb{N}\}$ of integers, let

$$\text{Weyl}(S, K) = \{x \in K : \text{ the sequence } \{a_n x : n \in \mathbb{N}\} = Sx \text{ is uniformly distributed in } K\}.$$

The classical setting of Kronecker-Weyl's theorem deals with the question of how large is the set $\text{Weyl}(S, K)$. When $K = \mathbb{T}$ (or more generally, when $K = \mathbb{T}^n$ for some $n \in \mathbb{N}^+$), the classical result of Weyl says that $\text{Weyl}(S, K)$ is a dense subset of K of (Haar) measure 1.

Definition 1.1. Let G be an abelian group and K a compact abelian group.

(i) For a one-to-one sequence $S = \{a_n : n \in \mathbb{N}\}$ in G, define

$$\mathbb{U}(S, K) = \{h \in \text{Hom}(G, K) : h(S) = \{h(a_n) : n \in \mathbb{N}\} \text{ is uniformly distributed in } K\}.$$

(ii) For an infinite subset S of an abelian group G, define

$$\mathbb{D}(S, K) = \{h \in \text{Hom}(G, K) : h(S) \text{ is dense in } K\}.$$

When $G = \mathbb{Z}$, the map $\theta : \text{Hom}(\mathbb{Z}, K) \to K$ defined by $\theta(h) = h(1)$ for every $h \in \text{Hom}(\mathbb{Z}, K)$, is a topological isomorphism between $\text{Hom}(\mathbb{Z}, K)$ and K such that $\theta(\mathbb{U}(S, K)) = \text{Weyl}(S, K)$ for every strictly increasing sequence $S = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{Z}$. In particular, the sets $\mathbb{U}(S, K)$ and $\text{Weyl}(S, K)$ have the same Borel complexity and are simultaneously dense in $\text{Hom}(\mathbb{Z}, K)$ and K, respectively. This observation allows us to identify the group K with the homomorphism group $\text{Hom}(\mathbb{Z}, K)$ and to focus our attention on the set $\mathbb{U}(S, K)$ instead of the set $\text{Weyl}(S, K)$.

Since uniformly distributed sequences in K are dense, $\mathbb{U}(S, K) \subseteq \mathbb{D}(S, K)$ for every one-to-one sequence $S = \{a_n : n \in \mathbb{N}\}$ in G. In fact, a certain converse also holds: If S is a countably infinite subset of G such that $h(S)$ is dense in K, then one can always find a one-to-one enumeration $S = \{a_n : n \in \mathbb{N}\}$ of the set S such that the sequence $\{h(a_n) : n \in \mathbb{N}\}$ becomes uniformly distributed in K. In other words, *it is the density in K that remains from a uniformly distributed sequence in K after forgetting its enumeration.* This allows us to consider $\mathbb{D}(S, K)$ as a natural *topological* counterpart of the set $\mathbb{U}(S, K)$.

There are other reasons why we prefer the set $\mathbb{D}(S, K)$ to the set $\mathbb{U}(S, K)$. Indeed, unlike the classical case of the integers $\mathbb{Z}$, *a priori* there is no natural order on an arbitrary abelian group G that allows us to index elements of its countably infinite subset S. The second reason is that the assignment $S \mapsto \mathbb{D}(S, K)$ is monotone (that is, $S \subseteq S' \subseteq G$ implies $\mathbb{D}(S, K) \subseteq \mathbb{D}(S', K)$), while there is no natural way to make the assignment $S \mapsto \mathbb{U}(S, K)$ monotone. At last but not least, $\mathbb{D}(S, K)$ has nicer descriptive properties than $\mathbb{U}(S, K)$; see Proposition 10.1.

To keep closer to the classical situation, we shall focus our attention on the case when K is a power $\mathbb{T}^\kappa$ of the torus group $\mathbb{T}$. Observe that $\text{Hom}(G, \mathbb{T})$ coincides with the Pontryagin dual group

$\widehat{G}$ of G, and $\mathrm{Hom}(G, \mathbb{T}^\kappa)$ is naturally isomorphic to the group $\mathrm{Hom}(G, \mathbb{T})^\kappa \cong \widehat{G}^\kappa$, for every cardinal κ. In the future we will always identify $\mathrm{Hom}(G, \mathbb{T}^\kappa)$ with $\widehat{G}^\kappa$.

Assume that τ and κ are cardinals such that $\tau \leq \kappa$, and let $q : \widehat{G}^\kappa \to \widehat{G}^\tau$ be the natural projection. Clearly, $q(\mathbb{D}(S, \mathbb{T}^\kappa)) \subseteq \mathbb{D}(S, \mathbb{T}^\tau)$ for every subset S of an abelian group G. In other words, the bigger the cardinal κ, the "harder" it is to send a given set S by a homomorphism $h : G \to \mathbb{T}^\kappa$ into a dense subset $h(S)$ of $\mathbb{T}^\kappa$, and so the "thinner" is the set $\mathbb{D}(S, \mathbb{T}^\kappa)$. In fact, there is a natural limit: If S is countable, then $\mathbb{D}(S, \mathbb{T}^\kappa) \neq \emptyset$ implies that $\mathbb{T}^\kappa$ is separable, which yields $\kappa \leq \mathfrak{c}$ by [15, Theorem 2.3.15]. Therefore, the case $\kappa = \mathfrak{c}$ represents the strongest possible version of any positive result that ensures $\mathbb{D}(S, \mathbb{T}^\kappa) \neq \emptyset$. This explains why $\mathbb{D}(S, \mathbb{T}^\mathfrak{c})$ appears in items (ii) and (iv) of Theorem 3.1, which is our principal result. This corollary characterizes subsets S of abelian groups G such that $\mathbb{D}(S, \mathbb{T}^\mathfrak{c}) \neq \emptyset$ and, moreover, it demonstrates that a mere non-emptiness of the set $\mathbb{D}(S, \mathbb{T}^\mathfrak{c})$ automatically guarantees that this set is "rather big" in $\mathrm{Hom}(G, \mathbb{T}^\mathfrak{c}) = \widehat{G}^\mathfrak{c}$. The previous discussion allows us to view Theorem 3.1 as an *extreme topological version* of the Kronecker-Weyl's theorem for arbitrary subsets of abelian groups.

The manuscript is organized as follows. In Section 2 we collect basic properties of the family $\mathfrak{T}_n(G)$ of almost n-torsion subsets of an abelian group G ($n \in \mathbb{N}$). All major results in this paper are corollaries of Theorem 2.4 whose proof is postponed until Section 9. Section 3 contains straightforward corollaries of this general theorem. In particular, we show that an abelian group G admits a dense monomorphism in $\mathbb{T}^\mathfrak{c}$ precisely when $|G| \leq 2^\mathfrak{c}$ and G is not bounded (Corollary 3.3). Applications to group actions (Corollary 3.4) and discrete flows (Corollary 3.5) on $\mathbb{T}^\mathfrak{c}$ are also given. In Section 4 we apply Theorem 2.4 to the problem of existence of precompact group topologies on an abelian group G realizing simultaneously the Zariski closure of countably many subsets of G (Theorem 4.1). In Section 5 we make a significant contribution to a problem of Markov, going back to 1944, asking for an algebraic description of potentially dense sets in groups. Corollary 5.2 completely solves this problem for countable subsets of abelian groups, while Corollary 5.1 gives a complete description of potentially dense subsets of abelian groups of size at most $2^\mathfrak{c}$. Even a particular case of this description (given in Corollary 5.3) solves a problem of Tkachenko and Yaschenko [23, Problem 6.6]. In Section 6 we apply our principal result to Bohr topologies and Bohr compactifications of abelian groups. In particular, we offer as a corollary an easy direct proof of classical results of Flor (Corollary 6.2) and Glicksberg (Corollary 6.3). Sections 7 and 8 develop tools necessary for the proof of the main result (Theorem 2.4) that is carried out in Section 9. The last Section 10 deals with Borel complexity of sets $\mathbb{D}(S, K)$ and $\mathbb{U}(S, K)$.

2. Main theorem

We say that $d \in \mathbb{N}$ is a *proper divisor of $n \in \mathbb{N}$* provided that $d \notin \{0, n\}$ and $dm = n$ for some $m \in \mathbb{N}$. Note that, according to our definition, each $d \in \mathbb{N}^+$ is a proper divisor of 0.

Definition 2.1. Let G be an abelian group.

(i) Let $n \in \mathbb{N}$. Following [7], we say that a countably infinite subset S of an abelian group G is *almost n-torsion* in G if $S \subseteq G[n]$ and the set $\{x \in S : dx = g\}$ is finite for each $g \in G$ and every proper divisor d of n.

(ii) For $n \in \mathbb{N}$, let $\mathfrak{T}_n(G)$ denote the family of all almost n-torsion sets in G.

(iii) Define $\mathfrak{T}(G) = \bigcup \{\mathfrak{T}_n(G) : n \in \mathbb{N}\}$.

The notion of an almost n-torsion set was introduced first in [12, Definition 3.3] under a different name and split into two cases; see [7, Remark 4.2] for an extended comparison between this terminology and the one proposed in [12]. Almost n-torsion sets found applications in [7, 11, 12, 23].

In order to clarify Definition 2.1 and to facilitate future references, we collect basic properties of almost n-torsion sets in our next remark.

Remark 2.2. Let G be an abelian group.

  (i) $\mathfrak{T}_1(G) = \emptyset$.
  (ii) $\mathfrak{T}_n(G) \cap \mathfrak{T}_m(G) = \emptyset$ for distinct $m, n \in \mathbb{N}$.
  (iii) Each family $\mathfrak{T}_n(G)$ is closed under taking infinite subsets, and so $\mathfrak{T}(G)$ has the same property.
  (iv) If H is a subgroup of G, then $\mathfrak{T}_n(H) = \{S \in \mathfrak{T}_n(G) : S \subseteq H\}$ for every $n \in \mathbb{N}$; see [7, Lemma 4.4]. In particular, whether a set S is almost n-torsion in G depends only on the subgroup of G generated by S.
  (v) If $S \in \mathfrak{T}_n(G)$ for some $n \in \mathbb{N}$, then the set $\{\pi \in \widehat{G} : \pi(S) \text{ is dense in } \mathbb{T}[n]\}$ is dense in $\widehat{G}$ (see [12, Lemma 3.7] for $n \geq 2$ and [23, Lemma 3.3] for $n = 0$). In particular, $\mathbb{D}(S, \mathbb{T})$ is dense in $\widehat{G}$ for every $S \in \mathfrak{T}_0(G)$.
  (vi) Every infinite subset X of G contains a set of the form $g + S$, where $g \in G$ and $S \in \mathfrak{T}_n(G)$ for some $n \in \mathbb{N}$ [12, Lemma 3.6].

Item (v) of this remark explains why almost n-torsion sets appear prominently in Theorems 2.4 and 3.1, as well as in Corollary 3.2.

Notation 2.3. For an abelian group G and $E \in \mathfrak{T}(G)$, we use $\mathfrak{n}_E$ to denote the unique integer $n \in \mathbb{N}$ such that $E \in \mathfrak{T}_n(G)$. (The uniqueness of such n follows from Remark 2.2(ii).)

Recall that a space X has the *Baire property* if $\bigcap\{U_n : n \in \mathbb{N}\}$ is dense in X for every sequence $\{U_n : n \in \mathbb{N}\}$ of dense open subsets of X.

All major results in this paper are corollaries of a single general theorem:

Theorem 2.4. *For an abelian group G and a countable subfamily $\mathscr{E} \neq \emptyset$ of $\mathfrak{T}(G)$, define*

(2) $$\Sigma_{G,\mathscr{E}} = \{\sigma \in \mathrm{Hom}(G, \mathbb{T}^{\mathfrak{c}}) : \sigma(E) \text{ is dense in } \mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c}} \text{ for every } E \in \mathscr{E}\}.$$

Then:

  (i) *$\Sigma_{G,\mathscr{E}}$ is a dense subset of $\mathrm{Hom}(G, \mathbb{T}^{\mathfrak{c}}) = \widehat{G}^{\mathfrak{c}}$ having the Baire property;*
  (ii) *$\Sigma_{G,\mathscr{E}} \cap \mathrm{Mono}(G, \mathbb{T}^{\mathfrak{c}})$ is a dense subset of $\widehat{G}^{\mathfrak{c}}$ having the Baire property if and only if $|G| \leq 2^{\mathfrak{c}}$.*

The proof of this theorem is postponed until Section 9.

Remark 2.5. The statement of Theorem 2.4 appears to be the best possible.

  (i) One cannot strengthen the conclusion of Theorem 2.4 by replacing $\mathfrak{c}$ in it with a cardinal $\kappa > \mathfrak{c}$. Indeed, if $E \in \mathscr{E}$, then $\mathbb{T}[\mathfrak{n}_E]^{\kappa}$ must be separable, and since $\mathfrak{n}_E \neq 1$ by Remark 2.2(i), this implies $\kappa \leq \mathfrak{c}$; see, for example, [15, Theorem 2.3.15].
  (ii) One cannot strengthen the statement of Theorem 2.4 by increasing the size of the family $\mathscr{E}$; see Remark 4.2(ii).
  (iii) One cannot strengthen the Baire property of $\Sigma_{G,\mathscr{E}}$ to requiring $\Sigma_{G,\mathscr{E}}$ to be a dense G_δ-subset of $\widehat{G}^{\mathfrak{c}}$. In fact, $\Sigma_{G,\mathscr{E}}$ does not even contain any non-empty G_δ-subset of $\widehat{G}^{\mathfrak{c}}$; see Remark 10.3(ii).

Remark 2.6. Let X be a dense subspace of a space Y.

  (i) X has the Baire property if and only if $X \cap W \cap \bigcap \mathscr{U} \neq \emptyset$ whenever W is a non-empty open subset of X and $\mathscr{U}$ is a countable family of open dense subsets of Y.
  (ii) If X is a dense subspace of Y having the Baire property, then so is every space Z satisfying $X \subseteq Z \subseteq Y$.

3. A KRONECKER-WEYL THEOREM FOR SUBSETS OF ABELIAN GROUPS

We start by providing a convenient reformulation of Definition 2.1 in the case $n = 0$: A subset E of an abelian group G is almost 0-torsion if and only if E is countably infinite, but $E \cap (g + G[k])$ is finite whenever $g \in G$ and $k \in \mathbb{N}^+$.

The next theorem is the principal result of this paper.

Theorem 3.1. *For a subset S of an abelian group G, the following conditions are equivalent:*

  (i) $\mathbb{D}(S, \mathbb{T}^\kappa) \neq \emptyset$ *for some cardinal $\kappa \geq 1$,*
  (ii) $\mathbb{D}(S, \mathbb{T}^{\mathfrak{c}})$ *is a dense subset of $\widehat{G}^{\mathfrak{c}}$ having the Baire property,*
  (iii) *S contains an almost 0-torsion set.*

Furthermore, if one additionally assumes that $|G| \leq 2^{\mathfrak{c}}$, then the following item can be added to the list of equivalent conditions (i)–(iii):

  (iv) $\mathbb{D}(S, \mathbb{T}^{\mathfrak{c}}) \cap \mathrm{Mono}(G, \mathbb{T}^{\mathfrak{c}})$ *is a dense subset of $\widehat{G}^{\mathfrak{c}}$ having the Baire property.*

Proof. The implication (ii)→(i) is clear.

(i)→(iii) Let κ be a cardinal from (i). Fix $\sigma \in \mathbb{D}(S, \mathbb{T}^\kappa)$. Since $\sigma(S)$ is dense in $\mathbb{T}^\kappa$, for every $n \in \mathbb{N}$ the set $n\sigma(S)$ must be dense in $n\mathbb{T}^\kappa = \mathbb{T}^\kappa$. In particular, nS must be infinite for every $n \in \mathbb{N}^+$. Then S contains an almost 0-torsion subset by [11, Proposition 5.11].

The implication (iv)→(ii) follows Remark 2.6(ii).

(iii)→(ii) Let $E \subseteq S$ be an almost 0-torsion set. Define $\mathscr{E} = \{E\}$. Since $\mathbb{T}[0] = \mathbb{T}$, (2) yields $\Sigma_{G,\mathscr{E}} \subseteq \mathbb{D}(S, \mathbb{T}^{\mathfrak{c}})$. From this and Theorem 2.4(i), we get (ii).

(iii)→(iv) Assume now that $|G| \leq 2^{\mathfrak{c}}$. Let E and $\mathscr{E}$ be as in the proof of the implication (iii)→(ii). Since $\Sigma_{G,\mathscr{E}} \cap \mathrm{Mono}(G, \mathbb{T}^{\mathfrak{c}}) \subseteq \mathbb{D}(S, \mathbb{T}^{\mathfrak{c}}) \cap \mathrm{Mono}(G, \mathbb{T}^{\mathfrak{c}})$, and the former set is a dense subset of $\widehat{G}^{\mathfrak{c}}$ having the Baire property by Theorem 2.4(ii), so is the latter set; see Remark 2.6(ii). $\square$

One cannot strengthen items (ii) or (iv) of Theorem 3.1 by requiring $\mathbb{D}(S, \mathbb{T}^{\mathfrak{c}})$ to be a dense G_δ-subset of $\widehat{G}^{\mathfrak{c}}$. Indeed, we shall show in Remark 10.3 that $\mathbb{D}(S, \mathbb{T}^{\mathfrak{c}})$ does not even contain any non-empty G_δ-subset of $\widehat{G}^{\mathfrak{c}}$.

Corollary 3.2. *For a subset S of an abelian group G, the following conditions are equivalent:*

  (i) *there exists a homomorphism $\pi : G \to \mathbb{T}^{\mathfrak{c}}$ such that $\pi(S)$ is dense in $\mathbb{T}^{\mathfrak{c}}$,*
  (ii) *S contains an almost 0-torsion set.*

Furthermore, if one additionally assumes that $|G| \leq 2^{\mathfrak{c}}$, then the following item can be added to the list of equivalent conditions (i) and (ii):

  (iii) *there exists a monomorphism $\pi : G \to \mathbb{T}^{\mathfrak{c}}$ such that $\pi(S)$ is dense in $\mathbb{T}^{\mathfrak{c}}$.*

The version of this corollary for homomorphisms (monomorphisms) into $\mathbb{T}^\kappa$ for cardinals $\kappa > \mathfrak{c}$ is proved in our paper [8].

According to the classical Kronecker theorem, for every n-tuple $(\alpha_1, \ldots, \alpha_n)$ of real numbers the cyclic subgroup of $\mathbb{T}^n$ generated by $(\alpha_1, \ldots, \alpha_n)$ is dense in $\mathbb{T}^n$ if and only if $1, \alpha_1, \ldots, \alpha_n$ are rationally independent. This implies the well-known fact that $\mathbb{T}^{\mathfrak{c}}$ contains a dense cyclic subgroup $\mathbb{Z}$. In our next corollary we describe the abelian groups that admit a dense embedding into $\mathbb{T}^{\mathfrak{c}}$:

Corollary 3.3. *An abelian group G is isomorphic to a dense subgroup of $\mathbb{T}^{\mathfrak{c}}$ if and only if G is not bounded and $|G| \leq 2^{\mathfrak{c}}$.*

Proof. Suppose that an abelian group G is not bounded and $|G| \leq 2^{\mathfrak{c}}$. By [11, Corollary 5.12], G contains an almost 0-torsion set S. Applying Corollary 3.2, we can find a monomorphism

$\pi : G \to \mathbb{T}^{\mathfrak{c}}$ such that $\pi(S)$ is dense in $\mathbb{T}^{\mathfrak{c}}$. Then $\pi(G)$ is dense in $\mathbb{T}^{\mathfrak{c}}$ as well. The reverse implication is clear. $\square$

Every homomorphism $\pi : G \to \mathbb{T}^{\mathfrak{c}}$ of an abelian group G defines the action $(g, x) \mapsto gx$ of G on $\mathbb{T}^{\mathfrak{c}}$ by $gx = \pi(g) + x$ for $g \in G$ and $x \in \mathbb{T}^{\mathfrak{c}}$. In this language Corollary 3.2 can be restated as follows.

Corollary 3.4. *Let S be a subset of an abelian group G such that S contains an almost 0-torsion set. Then there exists an action of G on $\mathbb{T}^{\mathfrak{c}}$ by homeomorphisms of $\mathbb{T}^{\mathfrak{c}}$ such that the "S-orbit" $\{gx : g \in S\}$ of each point $x \in \mathbb{T}^{\mathfrak{c}}$ is dense in $\mathbb{T}^{\mathfrak{c}}$.*

The following particular case of Corollary 3.4 seems to be new as well.

Corollary 3.5. *For every infinite subset S of $\mathbb{Z}$, there exists a translation $f : \mathbb{T}^{\mathfrak{c}} \to \mathbb{T}^{\mathfrak{c}}$ of the group $\mathbb{T}^{\mathfrak{c}}$ such that the "S-orbit" $\{f^n(x) : n \in S\}$ of each point $x \in \mathbb{T}^{\mathfrak{c}}$ is dense in $\mathbb{T}^{\mathfrak{c}}$.*

Proof. Follows from Corollary 3.4 and the fact that every infinite subset of $\mathbb{Z}$ is almost 0-torsion. $\square$

The the best of our knowledge, the following application gives a new contribution to diophantine approximation:

Corollary 3.6. *For every infinite set S of integers, there exists an indexed set $\{x_\alpha : \alpha < \mathfrak{c}\} \subseteq [0, 1)$ (depending on S and) having the following property: If $\varepsilon > 0$, $k \in \mathbb{N}^+$, $y_1, \ldots, y_k \in \mathbb{R}$ and $\alpha_1 < \mathfrak{c}, \ldots, \alpha_k < \mathfrak{c}$, then one can find $s \in S$ and $n_1, \ldots, n_k \in \mathbb{Z}$ such that $|sx_{\alpha_j} - y_j - n_j| < \varepsilon$ for every $j = 1, 2, \ldots, k$.*

Proof. Apply Corollary 3.2 to $G = \mathbb{Z}$ to find a monomorphism $\pi : \mathbb{Z} \to \mathbb{T}^{\mathfrak{c}}$ such that $\pi(S)$ is dense in $\mathbb{T}^{\mathfrak{c}}$. Let $\pi(1) = \{t_\alpha\}_{\alpha < \mathfrak{c}} \in \mathbb{T}^{\mathfrak{c}}$. For every $\alpha < \mathfrak{c}$, choose $x_\alpha \in [0, 1)$ such that $\psi(x_\alpha) = t_\alpha$, where $\psi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{T}$ is the natural quotient homomorphism. Then $\{x_\alpha : \alpha < \mathfrak{c}\}$ has the desired properties. $\square$

It should be noted that a much weaker version of Corollary 3.6, with $\mathfrak{c}$ replaced by ω, follows from results of [23].

4. Realization problem for the Zariski closure

Let G be an abelian group. According to Markov [21], a set of the form $g + G[n]$, for a suitable $g \in G$ and $n \in \mathbb{N}$, is called an *elementary algebraic* subset of G, and arbitrary intersections of finite unions of elementary algebraic subsets of G are called *algebraic* subsets of G. One can easily see that the family of all algebraic subsets of G is closed under finite unions and arbitrary intersections, and contains G and all finite subsets of G; thus, it can be taken as the family of closed sets of a unique T_1 topology $\mathfrak{Z}_G$ on G. Markov [20, 21] defined the *algebraic closure* of a subset X of G as the intersection of all algebraic subsets of G containing X, i.e., the smallest algebraic set that contains X. This definition satisfies the conditions necessary for introducing a topological closure operator on G. Since a topology on a set is uniquely determined by its closure operator, it is fair to say that Markov was the first to (implicitly) define the topology $\mathfrak{Z}_G$, though he did not name it. To the best of our knowledge, the first name for this topology appeared explicitly in print in a 1977 paper by Bryant [3], who called it a *verbal topology* of G. In a more recent series of papers beginning with [1], Baumslag, Myasnikov and Remeslennikov have developed algebraic geometry over an abstract group G. In an analogy with the Zariski topology from algebraic geometry, they introduced the Zariski topology on the finite powers G^n of a group G. In the particular case when $n = 1$, this topology coincides with the verbal topology of Bryant. For this reason, the topology $\mathfrak{Z}_G$ is also called the *Zariski topology* of G in [10, 9]. The topology $\mathfrak{Z}_G$ is Noetherian, and so compact.

While $\mathfrak{Z}_G$ is a T_1 topology, it is Hausdorff only when G is finite [11]. A comprehensive study of the Zariski topology is carried out in [11].

Given a topology $\mathcal{T}$ on G, we denote by $\mathrm{Cl}_{\mathcal{T}}(X)$ the $\mathcal{T}$-closure of a set $X \subseteq G$. We call $\mathrm{Cl}_{\mathfrak{Z}_G}(X)$ the *Zariski closure* of X in G. (Thus, $\mathrm{Cl}_{\mathfrak{Z}_G}(X)$ is the algebraic closure of X in the terminology of Markov [20, 21]). For every Hausdorff group topology $\mathcal{T}$ on G, one has $\mathfrak{Z}_G \subseteq \mathcal{T}$, and therefore, $\mathrm{Cl}_{\mathcal{T}}(X) \subseteq \mathrm{Cl}_{\mathfrak{Z}_G}(X)$ for each set $X \subseteq G$. This inclusion naturally leads to the following *realization problem for the Zariski closure*: For a given set $X \subseteq G$, can one always find a Hausdorff group topology $\mathcal{T}$ on G such that $\mathrm{Cl}_{\mathcal{T}}(X) = \mathrm{Cl}_{\mathfrak{Z}_G}(X)$? This problem was first considered by Markov in [21], who proved that for every subset X of a countable group G, there exists a metric group topology $\mathcal{T}$ on G such that $\mathrm{Cl}_{\mathcal{T}}(X) = \mathrm{Cl}_{\mathfrak{Z}_G}(X)$. We make the following contribution to this general problem in the abelian case:

Theorem 4.1. *Let G be an abelian group of size at most $2^{\mathfrak{c}}$, and let $\mathscr{X}$ be a countable family of subsets of G. Then there exists a precompact Hausdorff group topology $\mathcal{T}$ on G such that the $\mathcal{T}$-closure of each $X \in \mathscr{X}$ coincides with its Zariski closure.*

Proof. According to [11, Theorem 7.1], for every $X \in \mathscr{X}$ there exist a finite family $\mathscr{E}_X \subseteq \mathfrak{T}(G)$ and finite sets $F_X \subseteq G$ and $\{h_{E,X} : E \in \mathscr{E}_X\} \subseteq G$ such that

$$(3) \qquad F_X \cup \bigcup_{E \in \mathscr{E}_X} h_{E,X} + E \subseteq X \quad \text{and} \quad \mathrm{Cl}_{\mathfrak{Z}_G}(X) = F_X \cup \bigcup_{E \in \mathscr{E}_X} h_{E,X} + G[\mathfrak{n}_E].$$

Applying Theorem 2.4(ii) to G and $\mathscr{E} = \bigcup\{\mathscr{E}_X : X \in \mathscr{X}\}$, we can find a monomorphism $\sigma : G \to \mathbb{T}^{\mathfrak{c}}$ such that $\sigma \in \Sigma_{G,\mathscr{E}}$. Without loss of generality, we shall identify G with the subgroup $\sigma(G)$ of $\mathbb{T}^{\mathfrak{c}}$. That is, we shall assume that $G \subseteq \mathbb{T}^{\mathfrak{c}}$ and σ is the identity map. Under these assumptions, from (2) we conclude that each $E \in \mathscr{E}$ is dense in $\mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c}}$. We claim that the precompact group topology $\mathcal{T}$ induced on G by the topology of $\mathbb{T}^{\mathfrak{c}}$ has the desired property.

Indeed, let $X \in \mathscr{X}$. Fix $E \in \mathscr{E}_X$. Since $\mathscr{E}_X \subseteq \mathscr{E}$, E is dense in $\mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c}}$. Since $E \subseteq G[\mathfrak{n}_E] = G \cap \mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c}}$, we conclude that E is $\mathcal{T}$-dense in the $\mathcal{T}$-closed set $G[\mathfrak{n}_E]$. Thus, $\mathrm{Cl}_{\mathcal{T}}(E) = G[\mathfrak{n}_E]$. We showed that $\mathrm{Cl}_{\mathcal{T}}(E) = G[\mathfrak{n}_E]$ for every $E \in \mathscr{E}_X$. From this and (3), we obtain

$$\mathrm{Cl}_{\mathfrak{Z}_G}(X) = F_X \cup \bigcup_{E \in \mathscr{E}_X} h_{E,X} + G[\mathfrak{n}_E] = F_X \cup \bigcup_{E \in \mathscr{E}_X} h_{E,X} + \mathrm{Cl}_{\mathcal{T}}(E) =$$

$$F_X \cup \bigcup_{E \in \mathscr{E}_X} \mathrm{Cl}_{\mathcal{T}}(h_{E,X} + E) = \mathrm{Cl}_{\mathcal{T}}\left(F_X \cup \bigcup_{E \in \mathscr{E}_X} h_{E,X} + E \right) \subseteq \mathrm{Cl}_{\mathcal{T}}(X).$$

The reverse inclusion $\mathrm{Cl}_{\mathcal{T}}(X) \subseteq \mathrm{Cl}_{\mathfrak{Z}_G}(X)$ follows from the inclusion $\mathfrak{Z}_G \subseteq \mathcal{T}$. $\square$

Our next remark shows that one cannot increase the size of the family $\mathscr{X}$ in Theorem 4.1.

Remark 4.2. (i) Theorem 4.1 fails for $G = \mathbb{Z}$ and the family of all subsets of G taken as $\mathscr{X}$. Indeed, if our theorem were true in such a case, then the Zariski topology on G would coincide with $\mathcal{T}$. Since the Zariski topology for $\mathbb{Z}$ is co-finite (and thus, non-Hausdorff), we get a contradiction.

(ii) Theorem 2.4 fails for $G = \mathbb{Z}$ and the family $\mathscr{E}$ of all infinite subsets of G. (Note that $\mathscr{E} = \mathfrak{T}_0(G) \subseteq \mathfrak{T}(G)$.) Indeed, a careful analysis of the proof of Theorem 4.1 shows that, if Theorem 2.4 would hold for such G and $\mathscr{E}$, then Theorem 4.1 would also hold for G and $\mathscr{X}$ from item (i), giving a contradiction.

Corollary 4.3. *For an abelian group G, the following conditions are equivalent:*

(i) $|G| \leq 2^{\mathfrak{c}}$,

(ii) *for every subset X of G, there exists a precompact Hausdorff group topology $\mathcal{T}_X$ on G such that $\mathrm{Cl}_{\mathcal{T}_X}(X) = \mathrm{Cl}_{\mathfrak{Z}_G}(X)$,*

(iii) *for every countable family $\mathscr{X}$ of subsets of G, one can find a precompact Hausdorff group topology $\mathcal{T}_{\mathscr{X}}$ on G such that $\mathrm{Cl}_{\mathcal{T}_{\mathscr{X}}}(X) = \mathrm{Cl}_{\mathfrak{Z}_G}(X)$ for every $X \in \mathscr{X}$.*

Proof. The implication (i)→(iii) is proved in Theorem 4.1. The implication (iii)→(ii) is trivial. Let us prove the implication (ii)→(i). According to [11, Corollary 8.9], the topology $\mathfrak{Z}_G$ is separable, so there exists a countable set $X \subseteq G$ with $\mathrm{Cl}_{\mathfrak{Z}_G}(X) = G$. Applying item (ii) to this X, we can choose a (precompact) Hausdorff group topology $\mathcal{T}_X$ on G such that $\mathrm{Cl}_{\mathcal{T}_X}(G) = \mathrm{Cl}_{\mathfrak{Z}_G}(X) = G$. That is, $\mathcal{T}_X$ is separable. Since $\mathcal{T}_X$ is Hausdorff, this yields (i) by [22]; see also [15, Theorem 1.5.3]. □

A counterpart of this corollary, with the word "metric" added to both items (ii) and (iii), and the inequality in item (i) strengthened to $|G| \leq \mathfrak{c}$, is proved in our paper [11].

5. MARKOV'S POTENTIAL DENSITY

According to Markov [21], a subset X of a group G is *potentially dense* in G if G admits a Hausdorff group topology $\mathcal{T}$ such that X is $\mathcal{T}$-dense in G. The last section of Markov's paper [21] is exclusively dedicated to the following problem: *which subsets of a group G are potentially dense in G?* Markov showed that every infinite subset of $\mathbb{Z}$ is potentially dense in $\mathbb{Z}$ [21]. This was strengthened in [13, Lemma 5.2] by proving that every infinite subset of $\mathbb{Z}$ is dense in some precompact metric group topology on $\mathbb{Z}$. (The authors of [21] and [13] were apparently unaware that both these results easily follow from the uniform distribution theorem of Weyl [25].) Further progress was made by Tkachenko and Yaschenko [23], who proved the following theorem: If an abelian group G of size at most $\mathfrak{c}$ is either almost torsion-free or has exponent p for some prime number p, then every infinite subset of G is potentially dense in G. (According to [23], an abelian group G is *almost torsion-free* if the p-rank of G is finite for every prime number p.)

In [8], the authors resolved Markov's potential density problem for uncountable subsets of divisible and almost torsion-free abelian groups, among other classes. Our next two corollaries provide a solution to Markov's problem for arbitrary subsets of abelian groups of size at most $2^{\mathfrak{c}}$ and for countable subsets of arbitrary abelian groups, respectively.

Corollary 5.1. *Let X be a subset of an abelian group G such that $|G| \leq 2^{\mathfrak{c}}$. Then the following conditions are equivalent:*

(i) *X is potentially dense in G,*

(ii) *X is $\mathcal{T}$-dense in G for some precompact Hausdorff group topology $\mathcal{T}$ on G,*

(iii) *$\mathrm{Cl}_{\mathfrak{Z}_G}(X) = G$.*

Proof. The implication (iii)→(ii) follows from Corollary 4.3. Clearly, (ii) implies (i). To prove the implication (i)→(iii), let $\mathcal{T}$ be a Hausdorff group topology on G such that X is $\mathcal{T}$-dense in G. Then $G = \mathrm{Cl}_{\mathcal{T}}(X) \subseteq \mathrm{Cl}_{\mathfrak{Z}_G}(X) \subseteq G$, because $\mathfrak{Z}_G \subseteq \mathcal{T}$. Therefore, $\mathrm{Cl}_{\mathfrak{Z}_G}(X) = G$. □

Corollary 5.2. *For a countably infinite subset X of an abelian group G, the following conditions are equivalent:*

(i) *X is potentially dense in G,*

(ii) *there exists a precompact Hausdorff group topology $\mathcal{T}$ on G such that X is $\mathcal{T}$-dense in G,*

(iii) *$|G| \leq 2^{\mathfrak{c}}$ and $\mathrm{Cl}_{\mathfrak{Z}_G}(X) = G$.*

Proof. Assume (i). Then X is $\mathcal{T}$-dense in G for some Hausdorff group topology $\mathcal{T}$ on G, and so $\mathcal{T}$ is separable, which yields $|G| \leq 2^{\mathfrak{c}}$ by [22]; see also [15, Theorem 1.5.3]. The rest follows from Corollary 5.1. □

A counterpart of Corollaries 5.1 and 5.2, with the word "metric" added to their items (ii), and the condition on a group G strengthened to $|G| \leq \mathfrak{c}$, is proved in our paper [11].

We say that a subset X of an abelian group G is *Zariski dense* in G provided that $\mathrm{Cl}_{\mathfrak{Z}_G}(X) = G$. Items (iii) of Corollaries 5.1 and 5.2 make it important to characterize Zariski dense sets. One such "characterization" simply follows from the definition. A subset X of an abelian group G is Zariski dense in G provided that, if $k \in \mathbb{N}$, $g_1, g_2, \ldots, g_k \in G$, $n_1, n_2, \ldots, n_k \in \mathbb{N}$ and each $x \in X$ satisfies the equation $n_i x = g_i$ for some $i = 1, 2, \ldots, k$ (depending on x), then every $g \in G$ also satisfies some equation $n_j g = g_j$, for a suitable $j = 1, 2, \ldots, k$. A complete description in terms of almost n-torsion sets is given in [11]. If an abelian group G is not bounded, then a subset X of G is Zariski dense in G if and only if nX is infinite for every $n \in \mathbb{N}^+$ if and only if X contains an almost 0-torsion set [11, Theorem 7.4]. A subset X of an infinite bounded abelian group G is Zariski dense in G if and only if $g + X$ contains an almost m-torsion set for every $g \in G$, where m is the smallest positive integer such that mG is finite [11, Theorem 7.1].

As was mentioned in the text preceding Corollary 5.1, Tkachenko and Yaschenko proved in [23] that if $|G| \leq \mathfrak{c}$ and G is either almost torsion-free or has exponent p for some prime number p, then every infinite subset of G is potentially dense in G. In the same manuscript, the authors asked whether the restriction $|G| \leq \mathfrak{c}$ in their result can be weakened to $|G| \leq 2^{\mathfrak{c}}$ ([23, Problem 6.6]). As a special case of the above corollary, we obtain a positive solution to this problem.

Corollary 5.3. *For an abelian group G with $|G| \leq 2^{\mathfrak{c}}$, the following conditions are equivalent:*
  (i) *every infinite subset of G is potentially dense in G,*
  (ii) *every infinite subset of G is $\mathcal{T}$-dense in G for some precompact Hausdorff group topology $\mathcal{T}$ on G,*
  (iii) *G is either almost torsion-free or has exponent p for some prime number p,*
  (iv) *every $\mathfrak{Z}_G$-closed subset of G is finite.*

Proof. The equivalence of (iii) and (iv) is clear from the definition of the Zariski topology (and was observed already in [23]). Notice that (iv) is equivalent to having $\mathrm{Cl}_{\mathfrak{Z}_G}(X) = G$ for every infinite subset X of G. The rest follows from Corollary 5.2. $\square$

Note that if some countably infinite set is potentially dense in G, then $|G| \leq 2^{\mathfrak{c}}$ by the implication (i)$\rightarrow$(iii) of Corollary 5.2, so this cardinality restriction in Corollary 5.3 is necessary.

6. Applications to Bohr topologies and Bohr compactifications

Let G be an abelian group. The strongest precompact group topology on G is called the *Bohr topology* of G, and we use $G^{\#}$ to denote the group G equipped with its Bohr topology. The completion bG of $G^{\#}$ is a compact abelian group called the *Bohr compactification* of G. The terms Bohr topology and Bohr compactification have been chosen as a reward to Harald Bohr for his work [2] on almost periodic functions closely related to the Bohr compactification.

In this section, we illustrate the power of Theorem 2.4 by offering simple proofs of some well-known properties of Bohr topologies and Bohr compactifications of abelian groups. We start with a result of van Douwen; see [14, Theorem 1.1.3(a)].

Corollary 6.1. *Let G be an abelian group and X its infinite subset. Then the closure $\overline{X}$ of X in bG has size at least $2^{\mathfrak{c}}$.*

Proof. Let g and S be as in Remark 2.2(vi). Then $\mathfrak{n}_S \neq 1$ by Remark 2.2(i). Apply Theorem 2.4(i) with $\mathscr{E} = \{S\}$ to fix a homomorphism $\sigma : G \to \mathbb{T}^{\mathfrak{c}}$ such that $\sigma(S)$ is dense in $\mathbb{T}[\mathfrak{n}_S]^{\mathfrak{c}}$. Let $\pi : bG \to \mathbb{T}^{\mathfrak{c}}$ be the homomorphism extending σ. Note that $g + \overline{S} = \overline{g + S} \subseteq \overline{X}$, and

hence $\pi(g) + \pi(\overline{S}) = \pi(g + \overline{S}) \subseteq \pi(\overline{X})$. Since $\overline{S}$ is a closed subset of the compact space bG, it is compact, and so is $\pi(\overline{S})$. In particular, $\pi(\overline{S})$ is closed in $\mathbb{T}^{\mathfrak{c}}$. Since $\sigma(S) = \pi(S) \subseteq \pi(\overline{S})$, $\sigma(S)$ is dense in $\mathbb{T}[\mathfrak{n}_S]^{\mathfrak{c}}$ and $\pi(\overline{S})$ is closed in $\mathbb{T}^{\mathfrak{c}}$, we conclude that $\pi(\overline{S}) = \mathbb{T}[\mathfrak{n}_S]^{\mathfrak{c}}$. In particular, $|\overline{X}| \geq |g + \overline{S}| = |\overline{S}| \geq |\pi(\overline{S})| = |\mathbb{T}[\mathfrak{n}_S]^{\mathfrak{c}}| = 2^{\mathfrak{c}}$. $\qquad\square$

The above corollary immediately yields the well-known result of Flor [16]:

Corollary 6.2. *If G is an abelian group and bG is its Bohr compactification, then no sequence of points in G converges in bG.*

The well-known theorem of Glicksberg [18] also becomes an easy corollary:

Corollary 6.3. *An abelian group $G^{\#}$ with the Bohr topology has no infinite compact subsets.*

Proof. Assume that K is in infinite compact subset of $G^{\#}$. Choose a countable set $X \subseteq K$, and let H be the smallest subgroup of G containing X. Then H is closed in $G^{\#}$ [4, Lemma 2.1], and so $H \cap K$ is a closed subset of K. Therefore, $H \cap K$ is a countable compact subset of $G^{\#}$, and thus, of bG. Since $X \subseteq H \cap K$, it follows that $\overline{X} \subseteq \overline{H \cap K} = H \cap K$ and $|\overline{X}| \leq |H \cap K| \leq \omega$. Since X is infinite, this contradicts Corollary 6.1. $\qquad\square$

Recall that a space X is *pseudocompact* if every real-valued continuous function defined on X is bounded [19]. The following generalization of Glicksberg's theorem, due to Comfort and Trigos-Arrieta [5], can also be easily derived:

Corollary 6.4. *An abelian group $G^{\#}$ with the Bohr topology has no infinite pseudocompact subsets.*

Proof. Assume that X is an infinite pseudocompact subspace of $G^{\#}$. Take a countably infinite subset S of X. Let H be the divisible hull of G, and let H_1 be the smallest divisible subgroup of H containing S. Then H_1 is countable. Since H_1 is divisible, H_1 splits, i.e. there exists an abelian group N such that $H = H_1 \oplus N$; see [17]. Let $\varphi : H \to H_1$ be the natural projection. Since $\varphi : H^{\#} \to H_1^{\#}$ is continuous, $\varphi(X)$ is a pseudocompact subset of $H_1^{\#}$. Since $|\varphi(X)| \leq |H_1| \leq \omega$, $\varphi(X)$ is compact by [15, Theorems 3.11.1 and 3.11.12]. Thus, $\varphi(X)$ is finite by Corollary 6.3. Since $S = \varphi(S) \subseteq \varphi(X)$, the set S must be finite as well, giving a contradiction. $\qquad\square$

7. GENERAL PRELIMINARIES

Lemma 7.1. *If S is a subset of an abelian group H and O is an open subset of an abelian topological group K, then the set $P_{S,O} = \{\pi \in \mathrm{Hom}(H, K) : \pi(S) \cap O \neq \emptyset\}$ is open in $\mathrm{Hom}(H, K)$.*

Proof. Assume that $\pi_0 \in P_{S,O}$. Pick $x \in \pi_0^{-1}(O) \cap S$. Then $W = \{\pi \in \mathrm{Hom}(H, K) : \pi(x) \in O\}$ is an open subset of $\mathrm{Hom}(H, K)$ such that $\pi_0 \in W \subseteq P_{S,O}$. $\qquad\square$

Lemma 7.2. *If H is a countable abelian group and Y is a countably infinite set, then $\mathrm{Mono}(H, \mathbb{T}^Y)$ is a dense G_δ-subset of $\widehat{H}^Y$.*

Proof. For every $h \in H$, the set $V_h = \{\pi \in \widehat{H}^Y : \pi(h) \neq 0\}$ is open in $\widehat{H}^Y$ by Lemma 7.1, and so $\mathrm{Mono}(H, \mathbb{T}^Y) = \bigcap\{V_h : h \in H \setminus \{0\}\}$ is a G_δ-set in $\widehat{H}^Y$.

Let O be an arbitrary non-empty open subset of $\widehat{H}^Y$. There exists a non-empty open subset V of $\widehat{H}^F$ such that $V \times \widehat{H}^{Y \setminus F} \subseteq O$. Choose $\rho \in V$. Since $|H| = |Y \setminus F| = \omega$, there exists a monomorphism $\xi : H \to \mathbb{T}^{Y \setminus F}$. Then $\sigma = (\rho, \xi) \in V \times \widehat{H}^{Y \setminus F} \subseteq O$ is a monomorphism as well. This proves that $\sigma \in O \cap \mathrm{Mono}(H, \mathbb{T}^Y) \neq \emptyset$. Therefore, $\mathrm{Mono}(H, \mathbb{T}^Y)$ is dense in $\widehat{H}^Y$. $\qquad\square$

Given a subset Y of a set X and a subgroup H of an abelian group G, we define the map $q_{GH}^{XY} : \widehat{G}^X \to \widehat{H}^Y$ by $q_{GH}^{XY}\left(\{\chi_x\}_{x \in X}\right) = \{\chi_x \restriction_H\}_{x \in Y}$ for every $\{\chi_x\}_{x \in X} \in \widehat{G}^X$. Note that, after the natural identification, $\widehat{G}^X = \mathrm{Hom}(G, \mathbb{T}^X)$ becomes a closed subgroup of $\mathbb{T}^{X \times G}$, and the map q_{GH}^{XY} becomes the restriction to $\widehat{G}^X$ of the natural projection map $p_{GH}^{XY} : \mathbb{T}^{X \times G} \to \mathbb{T}^{Y \times H}$. In particular, the map q_{GH}^{XY} is continuous.

Lemma 7.3. *Assume that Z is a countable subset of an abelian group G, X is an infinite set and $\mathscr{V}$ is a countable family of open subsets of $\widehat{G}^X$. Then there exist a countable subgroup H of G containing Z, a countably infinite set $Y \subseteq X$ and a family $\mathscr{U} = \{U_V : V \in \mathscr{V}\}$ of open subsets of $\widehat{H}^Y$ such that $\left(q_{GH}^{XY}\right)^{-1}(U_V)$ is a dense subset of V for every $V \in \mathscr{V}$.*

Proof. Fix $V \in \mathscr{V}$. Let $\mathscr{B}_V$ be the family of basic open subsets B of the product $\mathbb{T}^{X \times G}$ such that $\emptyset \neq B \cap \widehat{G}^X \subseteq V$. We apply Zorn's lemma to select a maximal subfamily $\mathscr{W}_V$ of $\mathscr{B}_V$ consisting of pairwise disjoint subsets of V. Since $\widehat{G}^X$ is a compact group and every non-empty open subset of $\widehat{G}^X$ has a positive Haar measure, $\mathscr{W}_V$ must be countable. Therefore, $\mathscr{W} = \bigcup_{V \in \mathscr{V}} \mathscr{W}_V$ is a countable family of basic open subsets of $\mathbb{T}^{X \times G}$, so there exist countable sets $I \subseteq G$ and $J \subseteq X$ such that each element of $\mathscr{W}$ depends only on coordinates in $I \times J$; that is,

$$(4) \qquad W = \left(p_{GI}^{XJ}\right)^{-1}\left(p_{GI}^{XJ}(W)\right) \text{ for every } W \in \mathscr{W}.$$

Fix a countably infinite subset Y_0 of X. Then $Y = J \cup Y_0$ is countably infinite, and the subgroup H of G generated by $I \cup Z$ is countable as well. For typographical reasons, we let $p = p_{GH}^{XY}$ and $q = q_{GH}^{XY}$.

Let $V \in \mathscr{V}$ be arbitrary. The set $V' = \left(\bigcup \mathscr{W}_V\right) \cap \widehat{G}^X$ is dense in V by maximality of $\mathscr{W}_V$. Since $I \subseteq H$ and $J \subseteq Y$, from (4) it follows that $W = p^{-1}(p(W))$ for every $W \in \mathscr{W}$. Since $\mathscr{W}_V \subseteq \mathscr{W}$, this gives $\bigcup \mathscr{W}_V = p^{-1}(p(\bigcup \mathscr{W}_V))$, and so $V' = q^{-1}(q(V'))$. Since $q : \widehat{G}^X \to \widehat{H}^Y$ is a continuous surjective homomorphism defined on the compact group $\widehat{G}^X$, the map q is open. Since V' is an open subset of $\widehat{G}^X$, the set $U_V = q(V')$ is an open subset of $\widehat{H}^Y$. $\qquad \square$

8. Almost n-torsion sets and the powers of the dual group

Lemma 8.1. *Assume that H is an abelian group, $S \in \mathfrak{T}(H)$, Y is a non-empty set and O is an open subset of $\mathbb{T}^Y$ such that $O \cap \mathbb{T}[\mathfrak{n}_S]^Y \neq \emptyset$. Then*

$$(5) \qquad P_{S,O} = \{\pi \in \widehat{H}^Y : \pi(S) \cap O \neq \emptyset\}$$

is an open dense subset of $\widehat{H}^Y$.

Proof. The set $P_{S,O}$ is open in $\widehat{H}^Y = \mathrm{Hom}(H, \mathbb{T}^Y)$ by Lemma 7.1. Let us prove that $P_{S,O}$ is dense in $\widehat{H}^Y$. We start with the case of a finite Y.

Claim 1. $P_{S,O}$ is dense in $\widehat{H}^Y$ when Y is finite.

Proof. For $\mathfrak{n}_S \geq 2$, the density of $P_{S,O}$ in $\widehat{H}^Y$ follows from [12, Lemma 3.7]. By Remark 2.2(i), only the case $\mathfrak{n}_S = 0$ remains, so we shall assume now that $\mathfrak{n}_S = 0$; that is, $S \in \mathfrak{T}_0(H)$. The rest of the proof proceeds by induction on the size of the set Y. If $|Y| = 1$, then $P_{S,O}$ is dense in $\widehat{H}$ by [23, Lemma 3.3]; see also [12, Lemma 4.2]. Let $k \in \mathbb{N}^+$, and assume that we have already proved that the set P_{S^*,O^*} is dense in $\widehat{H}^{Y^*}$ whenever $S^* \in \mathfrak{T}_0(H)$, Y^* is a finite set with $1 \leq |Y^*| \leq k$ and O^* is a non-empty open subset of $\mathbb{T}^{Y^*}$. Suppose now that $S \in \mathfrak{T}_0(H)$, Y is a finite set of size $k+1$ and O is a non-empty open subset of $\mathbb{T}^Y$. Let V be a non-empty open subset of $\widehat{H}^Y$. It suffices to show that $V \cap P_{S,O} \neq \emptyset$.

Choose $y \in Y$ arbitrarily, and let $Y^* = Y \setminus \{y\}$. There exists an open subset O^* of $\mathbb{T}^{Y^*}$ and an open subset O' of $\mathbb{T}$ such that $O^* \times O' \subseteq O$. Similarly, there exist a non-empty open subset V^* of $\widehat{H}^{Y^*}$ and a non-empty open subset V' of $\widehat{H}$ such that $V^* \times V' \subseteq V$. Since $S \in \mathfrak{T}_0(H)$, from Remark 2.2(v) it follows that $\mathbb{D}(S, \mathbb{T})$ is dense in $\widehat{H}$, so we can pick $\pi' \in V'$ such that $\pi'(S)$ is dense in $\mathbb{T}$. Assume that $S^* = \{x \in S : \pi'(x) \in O'\}$ is finite. Then $O'' = O' \setminus \pi'(S^*)$ is a non-empty open subset of $\mathbb{T}$ such that $\pi'(S) \cap O'' = \emptyset$, in contradiction with density of $\pi'(S)$ in $\mathbb{T}$. Therefore, S^* must be infinite. Since $S^* \subseteq S \in \mathfrak{T}_0(H)$, Remark 2.2(iii) yields that $S^* \in \mathfrak{T}_0(H)$. Applying our inductive assumption to S^*, Y^* and O^*, we can choose $\pi^* \in V^* \cap P_{S^*, O^*}$. Then $\pi^*(S^*) \cap O^* \neq \emptyset$ by (5), so there exists $x \in S^* \subseteq S$ with $\pi^*(x) \in O^*$. Now $\pi = (\pi^*, \pi') \in V^* \times V' \subseteq V$ and $\pi(x) = (\pi^*(x), \pi'(x)) \in O^* \times O' \subseteq O$. Therefore, $\pi(S) \cap O \neq \emptyset$. We proved that $\pi \in V \cap P_{S,O} \neq \emptyset$. $\qquad\square$

Assume now that Y is infinite. Choose any $z = \{z_y\}_{y \in Y} \in O \cap \mathbb{T}[\mathfrak{n}_S]^Y$. Since O is open in $\mathbb{T}^Y$, there exist a non-empty finite subset F of Y and an open subset O_F of $\mathbb{T}^F$ such that $z \in O_F \times \mathbb{T}^{Y \setminus F} \subseteq O$. From this and (5), one gets $P_{S,O_F} \times \widehat{H}^{Y \setminus F} \subseteq P_{S,O}$. Since $z \in \mathbb{T}[\mathfrak{n}_S]^Y$, we have $z_F = \{z_y\}_{y \in F} \in O_F \cap \mathbb{T}[\mathfrak{n}_S]^F \neq \emptyset$. By Claim 1, P_{S,O_F} is dense in $\widehat{H}^F$, and so $P_{S,O_F} \times \widehat{H}^{Y \setminus F}$ is dense in $\widehat{H}^Y$. Therefore, $P_{S,O}$ must be dense in $\widehat{H}^Y$ as well. $\qquad\square$

The main idea of the proof of our next lemma comes from the classical proof of the Hewitt-Marczewski-Pondiczery theorem; see, for example, [15, Theorem 2.3.15]. Essentially, we produce a sophisticated adaptation of that proof to the power $\widehat{H}^{\mathfrak{c}}$ of the dual group $\widehat{H}$.

Lemma 8.2. *Let H be an abelian group and $\mathscr{S}$ a countable subfamily of $\mathfrak{T}(H)$. Then there exists a group homomorphism $\pi : H \to \mathbb{T}^{\mathfrak{c}}$ such that $\pi(S)$ is dense in $\mathbb{T}[\mathfrak{n}_S]^{\mathfrak{c}}$ for every $S \in \mathscr{S}$.*

Proof. In this proof only, it is beneficial for us to follow the common set-theoretic practice of identifying a natural number $m \in \mathbb{N}$ with the set $\{0, \dots, m-1\}$ of all its predecessors. In particular, $0 = \emptyset$ and $m = \{0, \dots, m-1\}$ for $m \in \mathbb{N}^+$.

Let $\mathscr{B} = \{B_l : l \in \mathbb{N}\}$ be a countable base of $\mathbb{T}$ such that $B_0 = \mathbb{T}$. Let $\mathscr{S} = \{S_j : j \in \mathbb{N}\}$ be an enumeration of $\mathscr{S}$. Define $n_j = \mathfrak{n}_{S_j}$ for every $j \in \mathbb{N}$. For every $m \in \mathbb{N}^+$, let

$$(6) \qquad O_\mu = \prod_{g \in 2^m} B_{\mu(g)} \quad \text{for every function } \mu : 2^m \to m, \text{ and}$$

$$(7) \qquad L_m = \left\{ (j, \mu) : j = 0, \dots, m-1, \mu : 2^m \to m \text{ is a function and } O_\mu \cap \mathbb{T}[n_j]^{2^m} \neq \emptyset \right\}.$$

By induction on $m \in \mathbb{N}$ we define a family $\{U_g : g \in 2^m\}$ of non-empty open subsets of $\widehat{H}$ with the following properties:

(i$_m$) if $m \in \mathbb{N}^+$, then $\overline{U_g} \subseteq U_{g \restriction m-1}$ for all $g \in 2^m$,

(ii$_m$) if $m \in \mathbb{N}^+$, then $\prod_{g \in 2^m} U_g \subseteq \bigcap \{P_{S_j, O_\mu} : (j, \mu) \in L_m\}$.

$U_\emptyset = \widehat{H}$ trivially satisfies (i$_0$) and (ii$_0$). Suppose that $m \in \mathbb{N}^+$, and for every $k < m$ we have already defined a family $\{U_g : g \in 2^k\}$ of non-empty open subsets of $\widehat{H}$ satisfying (i$_k$) and (ii$_k$). We are going to define a family $\{U_g : g \in 2^m\}$ of non-empty open subsets of $\widehat{H}$ satisfying (i$_m$) and (ii$_m$).

For each $g \in 2^m$, since $U_{g \restriction m-1} \neq \emptyset$ and the space $\widehat{H}$ is completely regular, we can choose a non-empty open subset V_g of $\widehat{H}$ with $\overline{V_g} \subseteq U_{g \restriction m-1}$. Since L_m is finite, applying Lemma 8.1 to $Y = 2^m$, we conclude that $P = \bigcap \{P_{S_j, O_\mu} : (j, \mu) \in L_m\}$ is an open dense subset of $\widehat{H}^{2^m}$. Since $V = \prod_{g \in 2^m} V_g$ is a non-empty open subset of $\widehat{H}^{2^m}$, so is $P \cap V$. Therefore, there exists a family $\{U_g : g \in 2^m\}$ of non-empty open subsets of $\widehat{H}$ such that $\prod_{g \in 2^m} U_g \subseteq P \cap V$. By our construction,

$U_g \subseteq V_g \subseteq \overline{V_g} \subseteq U_{g \restriction m-1}$ for all $g \in 2^m$, so (i_m) holds. Clearly, (ii_m) holds as well. This finishes the inductive construction.

Let $f \in 2^{\mathbb{N}}$. Since (i_m) holds for all $m \in \mathbb{N}$, we have

$$U_\emptyset = U_{f \restriction 0} \supseteq \overline{U_{f \restriction 1}} \supseteq U_{f \restriction 1} \supseteq \overline{U_{f \restriction 2}} \supseteq U_{f \restriction 2} \supseteq \cdots \supseteq \overline{U_{f \restriction m}} \supseteq U_{f \restriction m} \supseteq \overline{U_{f \restriction m+1}} \supseteq U_{f \restriction m+1} \supseteq \cdots .$$

Since all $U_{f \restriction m}$ are non-empty and $\widehat{H}$ is compact, this yields

$$(8) \qquad C_f = \bigcap_{m \in \mathbb{N}} U_{f \restriction m} = \bigcap_{m \in \mathbb{N}} \overline{U_{f \restriction m}} \neq \emptyset.$$

Therefore, there exists $\pi_f \in C_f$.

Define a homomorphism $\pi : H \to \mathbb{T}^{2^{\mathbb{N}}}$ by $\pi(x) = \{\pi_f(x)\}_{f \in 2^{\mathbb{N}}} \in \mathbb{T}^{2^{\mathbb{N}}}$ for all $x \in H$. Since $\mathbb{T}^{2^{\mathbb{N}}}$ and $\mathbb{T}^{\mathfrak{c}}$ are topologically isomorphic, it remains only to prove that $\pi(S)$ is dense in $\mathbb{T}[\mathfrak{n}_S]^{2^{\mathbb{N}}}$ for every $S \in \mathscr{S}$. Fix $S \in \mathscr{S}$. Then $S = S_j$ for some $j \in \mathbb{N}$. Since $\mathfrak{n}_S = \mathfrak{n}_{S_j} = n_j$, it suffices to prove that $\pi(S_j) \cap O \neq \emptyset$ for every open subset O of $\mathbb{T}^{2^{\mathbb{N}}}$ such that $O \cap \mathbb{T}[n_j]^{2^{\mathbb{N}}} \neq \emptyset$. Fix such an O. By the definition of the Tychonoff product topology, there exist a finite set $F \subseteq 2^{\mathbb{N}}$ and a set $\{l_f : f \in F\} \subseteq \mathbb{N}$, such that

(a) $\prod_{f \in F} B_{l_f} \cap \mathbb{T}[n_j]^F \neq \emptyset$, and
(b) $\prod_{f \in F} B_{l_f} \times \mathbb{T}^{2^{\mathbb{N}} \setminus F} \subseteq O$.

Claim 2. There exist $m \in \mathbb{N}$ and a function $\varphi : 2^m \to 2^{\mathbb{N}}$ such that:

(i) $m > \max\{l_f : f \in F\}$ and $m > j$,
(ii) $\varphi(g) \restriction m = g$ for all $g \in 2^m$,
(iii) $F \subseteq \varphi(2^m)$.

Proof. Choose $m \in \mathbb{N}^+$ such that $f \restriction m \neq f' \restriction m$ whenever $f, f' \in F$ and $f \neq f'$. Without loss of generality, we may assume that (i) holds. Let $G = \{f \restriction m : f \in F\}$. For $g \in G$, define $\varphi(g)$ to be the unique $f \in F$ with $g = f \restriction m$. For $g \in 2^m \setminus G$, let $\varphi(g)$ be an arbitrary $f \in 2^{\mathbb{N}}$ such that $f \restriction m = g$. Now (ii) and (iii) are satisfied. $\qquad\square$

Claim 2(i) allows us to define the function $\mu : 2^m \to m$ by letting $\mu(g) = l_{\varphi(g)}$ if $\varphi(g) \in F$ and $\mu(g) = 0$ otherwise, for every $g \in 2^m$.

Claim 3. $(j, \mu) \in L_m$.

Proof. Note that $j < m$ by Claim 2(i). According to (7), it remains only to check that $O_\mu \cap \mathbb{T}[n_j]^{2^m} \neq \emptyset$. By (6), to accomplish this, it suffices to show that $B_{\mu(g)} \cap \mathbb{T}[n_j] \neq \emptyset$ for every $g \in 2^m$. If $\varphi(g) \in F$, then $\mu(g) = l_{\varphi(g)}$, and so $B_{\mu(g)} \cap \mathbb{T}[n_j] = B_{l_{\varphi(g)}} \cap \mathbb{T}[n_j] \neq \emptyset$ by (a). If $\varphi(g) \notin F$, then $\mu(g) = 0$ and $B_{\mu(g)} \cap \mathbb{T}[n_j] = B_0 \cap \mathbb{T}[n_j] = \mathbb{T}[n_j] \neq \emptyset$, as $B_0 = \mathbb{T}$. $\qquad\square$

For each $g \in 2^m$, from Claim 2(ii) and (8), we get $\pi_{\varphi(g)} \in C_{\varphi(g)} \subseteq U_{\varphi(g) \restriction m} = U_g$. Therefore, $\{\pi_{\varphi(g)}\}_{g \in 2^m} \in \prod_{g \in 2^m} U_g$. From this, Claim 3 and (ii_m), it follows that $\{\pi_{\varphi(g)}\}_{g \in 2^m} \in P_{S_j, O_\mu}$. Combining this with (5) and (6), we can select $x \in S_j$ such that $\pi_{\varphi(g)}(x) \in B_{\mu(g)}$ whenever $g \in 2^m$.

Let $f \in F$ be arbitrary. It follows from items (ii) and (iii) of Claim 2 that $f = \varphi(g)$, where $g = f \restriction m \in 2^m$, and so $\pi_f(x) = \pi_{\varphi(g)}(x) \in B_{\mu(g)} = B_{l_{\varphi(g)}} = B_{l_f}$. From (b), we get $\pi(x) \in O$. Since $x \in S_j$, we obtain $\pi(x) \in \pi(S_j) \cap O \neq \emptyset$. $\qquad\square$

9. Proof of Theorem 2.4

We are going to carry out the proof of item (i) and the "if" part of item (ii) simultaneously. In order to do this, define $\Sigma_{G,\mathscr{E}}^{\flat} = \Sigma_{G,\mathscr{E}} \cap \mathrm{Mono}(G, \mathbb{T}^{\mathfrak{c}})$ if $|G| \leq 2^{\mathfrak{c}}$ and $\Sigma_{G,\mathscr{E}}^{\flat} = \Sigma_{G,\mathscr{E}}$ otherwise. Our goal is to prove that $\Sigma_{G,\mathscr{E}}^{\flat}$ is a dense subspace of $\widehat{G}^{\mathfrak{c}}$ having the Baire property. By Remark 2.6(i), in order to achieve this, it suffices to check that $\Sigma_{G,\mathscr{E}}^{\flat} \cap W \cap \bigcap \mathscr{V}^{*} \neq \emptyset$ whenever W is a non-empty open subset of $\widehat{G}^{\mathfrak{c}}$ and $\mathscr{V}^{*}$ is a countable family of open dense subsets of $\widehat{G}^{\mathfrak{c}}$.

First, we apply Lemma 7.3 to G, $Z = \bigcup \mathscr{E}$, $X = \mathfrak{c}$ and $\mathscr{V} = \mathscr{V}^{*} \cup \{W\}$ to choose H, Y and $\mathscr{U}$ as in the conclusion of this lemma. In particular, $\mathscr{U} = \{U_V : V \in \mathscr{V}\}$ is a family of open subsets of $\widehat{H}^{Y}$.

Claim 4. $U_W \neq \emptyset$ and U_V is a dense in $\widehat{H}^{Y}$ for every $V \in \mathscr{V}^{*}$.

Proof. Since the map q_{GH}^{XY} is continuous, from Lemma 7.3 it follows that U_V is dense in $q_{GH}^{XY}(V)$ for every $V \in \mathscr{V}$. In particular, U_W is dense in $q_{GH}^{XY}(W) \neq \emptyset$, so U_W must be non-empty. Since each $V \in \mathscr{V}^{*}$ is dense in $\widehat{G}^{X}$ and the map q_{GH}^{XY} is continuous, $q_{GH}^{XY}(V)$ must be dense in $q_{GH}^{XY}(\widehat{G}^{X}) = \widehat{H}^{Y}$. Hence, U_V is dense in H^{Y} for every $V \in \mathscr{V}^{*}$. $\qquad\square$

Let $\mathscr{B}$ be a countable base of $\mathbb{T}^{Y}$. Define

$$(9) \qquad D = \bigcap \left\{ P_{E,O} : E \in \mathscr{E}, O \in \mathscr{B}, O \cap \mathbb{T}[\mathfrak{n}_E]^{Y} \neq \emptyset \right\} \cap \mathrm{Mono}(H, \mathbb{T}^{Y}) \cap \bigcap_{V \in \mathscr{V}^{*}} U_V,$$

where $P_{E,O}$ are the sets defined in (5).

Claim 5. There exists $\theta \in D \cap U_W$.

Proof. If $E \in \mathscr{E}$, $O \in \mathscr{B}$ and $O \cap \mathbb{T}[\mathfrak{n}_E]^{Y} \neq \emptyset$, then $P_{E,O}$ is a dense open subset of $\widehat{H}^{Y}$ by Lemma 8.1. Furthermore, $\mathrm{Mono}(H, \mathbb{T}^{Y})$ is a dense G_δ-subset of $\widehat{H}^{Y}$ by Lemma 7.2. For every $V \in \mathscr{V}^{*}$, U_V is an open dense subset of $\widehat{H}^{Y}$ by Claim 4. Since $\mathscr{E}$, $\mathscr{B}$ and $\mathscr{V}^{*}$ are countable, D is an intersection of a countable family of open dense subsets of $\widehat{H}^{Y}$. Since $\widehat{H}^{Y}$ is compact, D must be dense in $\widehat{H}^{Y}$. Since U_W is a non-empty open subset of $\widehat{H}^{Y}$ by Claim 4, $D \cap U_W \neq \emptyset$. This allows us to choose $\theta \in D \cap U_W$. $\qquad\square$

For $E \in \mathscr{E}$ and $O \in \mathscr{B}$, let

$$(10) \qquad\qquad\qquad S_{E,O} = \{x \in E : \theta(x) \in O\}.$$

Since both $\mathscr{E}$ and $\mathscr{B}$ are countable, the set

$$(11) \qquad\qquad\qquad \mathscr{S} = \{S_{E,O} : E \in \mathscr{E}, O \in \mathscr{B}, |S_{E,O}| = \omega\}.$$

is countable as well.

Claim 6. There exists a homomorphism $\pi : H \to \mathbb{T}^{\mathfrak{c} \setminus Y}$ such that $\pi(S)$ is dense in $\mathbb{T}[\mathfrak{n}_S]^{\mathfrak{c} \setminus Y}$ for every $S \in \mathscr{S}$.

Proof. Let $S \in \mathscr{S}$. Then $S = S_{E,O} \subseteq E \subseteq \bigcup \mathscr{E} = Z \subseteq H$ for some $E \in \mathscr{E}$ and $O \in \mathscr{B}$. Since $\mathscr{E} \subseteq \mathfrak{T}(G)$ and S is infinite, from this and Remark 2.2(iii) we conclude that $\mathscr{S} \subseteq \mathfrak{T}(G)$. Since $S \subseteq H$ for every $S \in \mathscr{S}$, we also have $\mathscr{S} \subseteq \mathfrak{T}(H)$ by Remark 2.2(iv). Since Y is countable, $|\mathfrak{c} \setminus Y| = \mathfrak{c}$, and so $\mathbb{T}^{\mathfrak{c}}$ and $\mathbb{T}^{\mathfrak{c} \setminus Y}$ are topologically isomorphic. Now the conclusion follows from Lemma 8.2. $\qquad\square$

Since $\theta \in D \subseteq \mathrm{Mono}(H, \mathbb{T}^Y)$ by Claim 5 and (9), $\theta : H \to \mathbb{T}^Y$ is a monomorphism. Therefore, the map $\xi = (\theta, \pi) : H \to \mathbb{T}^Y \times \mathbb{T}^{\mathfrak{c} \backslash Y}$ is a monomorphism as well. Since $\mathbb{T}^{\mathfrak{c}}$ is divisible, there exists a homomorphism $\sigma : G \to \mathbb{T}^Y \times \mathbb{T}^{\mathfrak{c} \backslash Y} = \mathbb{T}^{\mathfrak{c}}$ extending ξ. Furthermore, if $|G| \le 2^{\mathfrak{c}}$, then [7, Lemma 3.17] allows us to find a monomorphism $\sigma : G \to \mathbb{T}^Y \times \mathbb{T}^{\mathfrak{c} \backslash Y} = \mathbb{T}^{\mathfrak{c}}$ extending ξ. Indeed, the assumptions of [7, Lemma 3.17] are satisfied, because $\mathbb{T}^{\mathfrak{c}}$ is divisible, $|G| \le 2^{\mathfrak{c}}$, $|H| = \omega < 2^{\mathfrak{c}}$, $|\mathbb{T}^{\mathfrak{c}}| = r(\mathbb{T}^{\mathfrak{c}}) = 2^{\mathfrak{c}}$ and $r_p(\mathbb{T}^{\mathfrak{c}}) = 2^{\mathfrak{c}}$ for all prime numbers p (see, for example, [6, Lemma 4.1] for computations of the ranks of $\mathbb{T}^{\mathfrak{c}}$).

Let us show that $\sigma \in \Sigma_{G,\mathscr{E}}^{\flat} \cap W \cap \bigcap \mathscr{V}^*$, thereby proving that $\Sigma_{G,\mathscr{E}}^{\flat} \cap W \cap \bigcap \mathscr{V}^* \ne \emptyset$. Note that

$$q_{GH}^{XY}(\sigma) = \theta \in D \cap U_W \subseteq \bigcap \{U_V : V \in \mathscr{V}^*\} \cap U_W = \bigcap \{U_V : V \in \mathscr{V}\}$$

by Claim 5 and (9), so

$$\sigma \in \left(q_{GH}^{XY}\right)^{-1}\left(\bigcap_{V \in \mathscr{V}} U_V\right) = \bigcap_{V \in \mathscr{V}} \left(q_{GH}^{XY}\right)^{-1}(U_V) \subseteq \bigcap_{V \in \mathscr{V}} V = \bigcap \mathscr{V} = W \cap \bigcap \mathscr{V}^*$$

by Lemma 7.3. Since $\sigma \in \mathrm{Mono}(G, \mathbb{T}^{\mathfrak{c}})$ when $|G| \le 2^{\mathfrak{c}}$, it remains only to check that $\sigma \in \Sigma_{G,\mathscr{E}}$. Fix $E \in \mathscr{E}$. Recalling (2), we need to prove that $\sigma(E)$ is dense in $\mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c}}$. Let O^* be an arbitrary open subset of $\mathbb{T}^{\mathfrak{c}}$ with $O^* \cap \mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c}} \ne \emptyset$. It suffices to prove that $\sigma(E) \cap O^* \ne \emptyset$. Since $\mathscr{B}$ is a base of $\mathbb{T}^Y$, there exist $O \in \mathscr{B}$ and an open subset V of $\mathbb{T}^{\mathfrak{c} \backslash Y}$ such that $O \times V \subseteq O^*$, $O \cap \mathbb{T}[\mathfrak{n}_E]^Y \ne \emptyset$ and $V \cap \mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c} \backslash Y} \ne \emptyset$.

Claim 7. $S_{E,O} \in \mathscr{S}$.

Proof. By (11), it suffices to show that the set $S_{E,O}$ is infinite. Recall that $\mathfrak{n}_E \ne 1$ by Remark 2.2(i). Since Y is infinite, the non-empty open subset $O \cap \mathbb{T}[\mathfrak{n}_E]^Y$ of $\mathbb{T}[\mathfrak{n}_E]^Y$ is infinite. Assume that $S_{E,O}$ is finite. Then $O' = O \backslash \theta(S_{E,O})$ is a non-empty open subset of $\mathbb{T}^Y$ with $O' \cap \mathbb{T}[\mathfrak{n}_E]^Y \ne \emptyset$. Since $\mathscr{B}$ is a base of $\mathbb{T}^Y$, we can choose $O'' \in \mathscr{B}$ such that $O'' \subseteq O'$ and $O'' \cap \mathbb{T}[\mathfrak{n}_E]^Y \ne \emptyset$. Then $\theta \in D \subseteq P_{E,O''}$ by (9), and from (5) we conclude that $\theta(x) \in O'' \subseteq O'$ for some $x \in E$. Since $O' \subseteq O$, from (10) it follows that $x \in S_{E,O}$, which yields $\theta(x) \in \theta(S_{E,O})$. Therefore, $\theta(x) \notin O \backslash \theta(S_{E,O}) = O'$, giving a contradiction. $\square$

Claim 8. $\mathfrak{n}_{S_{E,O}} = \mathfrak{n}_E$.

Proof. Being an infinite subset of the almost $\mathfrak{n}_E$-torsion set E, the set $S_{E,O}$ is also almost $\mathfrak{n}_E$-torsion by Remark 2.2(iii). Now the conclusion follows from Remark 2.2(ii). $\square$

The set $\pi(S_{E,O})$ is dense in $\mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c} \backslash Y}$ by Claims 6, 7 and 8. Since $V \cap \mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c} \backslash Y}$ is a non-empty open subset of $\mathbb{T}[\mathfrak{n}_E]^{\mathfrak{c} \backslash Y}$, there exists $x \in S_{E,O}$ with $\pi(x) \in V$. Therefore, $\theta(x) \in O$ by (10), and so $\xi(x) = (\theta(x), \pi(x)) \in O \times V \subseteq O^*$. Since $x \in S_{E,O} \subseteq E$, it follows that $\xi(x) \in \xi(E) \cap O^* \ne \emptyset$. Since $x \in E \subseteq Z \subseteq H$ and $\sigma \restriction_H = \xi$, we conclude that $\sigma(x) \in \sigma(E) \cap O^* \ne \emptyset$. This finishes the proof of item (i) and the "if" part of item (ii).

Assume now that $\Sigma_{G,\mathscr{E}} \cap \mathrm{Mono}(G, \mathbb{T}^{\mathfrak{c}})$ is dense in $\widehat{G}^{\mathfrak{c}}$, and choose $\sigma \in \Sigma_{G,\mathscr{E}} \cap \mathrm{Mono}(G, \mathbb{T}^{\mathfrak{c}})$. Since σ is a monomorphism, $|G| = |\sigma(G)| \le |\mathbb{T}^{\mathfrak{c}}| \le 2^{\mathfrak{c}}$. This proves the "only if" part of item (ii). $\square$

10. COMPARISON OF BOREL COMPLEXITY OF $\mathbb{D}(S, K)$ AND $\mathbb{U}(S, K)$

Proposition 10.1. *Let G be an abelian group and K a compact metric abelian group.*

   (i) *If S is a countably infinite subset of G, then $\mathbb{D}(S, K)$ is a G_δ-set in $\mathrm{Hom}(G, K)$.*
   (ii) *If $S = \{a_n : n \in \mathbb{N}\}$ is a one-to-one sequence in G, then $\mathbb{U}(S, K)$ is an $F_{\sigma\delta}$-set in $\mathrm{Hom}(G, K)$.*

Proof. (i) Fix a countable base $\mathscr{B}$ of K such that $\emptyset \notin \mathscr{B}$. For every $O \in \mathscr{B}$, the set $P_{S,O} = \{\pi \in \text{Hom}(G,K) : \pi(S) \cap O \neq \emptyset\}$ is open in $\text{Hom}(G,K)$ by Lemma 7.1 (applied to G instead of H). Since $\mathbb{D}(S,K) = \bigcap \{P_{S,O} : O \in \mathscr{B}\}$, it follows that $\mathbb{D}(S,K)$ is a G_δ-set in $\text{Hom}(G,K)$.

(ii) Since K is a compact metric group, the dual group $\widehat{K}$ is countable. Let $\{\chi_k : k \in \mathbb{N}\}$ be an enumeration of all non-trivial continuous $\mathbb{S}$-characters of K. For $k \in \mathbb{N}$ and $m, n \in \mathbb{N}^+$,

$$F_{k,m,n} = \left\{ \pi \in \text{Hom}(G,K) : \left| \frac{1}{n} \sum_{i=1}^{n} \chi_k(\pi(a_i)) \right| \leq \frac{1}{m} \right\}$$

is a closed subset of $\text{Hom}(G,K)$, and so $\bigcap_{n=j}^{\infty} F_{k,m,n}$ is also a closed subset of $\text{Hom}(G,K)$ for every $j \in \mathbb{N}^+$. Combining this with Weyl's criterion, we conclude that

$$\mathbb{U}(S,K) = \bigcap_{k \in \mathbb{N}} \bigcap_{m \in \mathbb{N}^+} \bigcup_{j \in \mathbb{N}^+} \left(\bigcap_{n=j}^{\infty} F_{k,m,n} \right)$$

is an $F_{\sigma\delta}$-subset of $\text{Hom}(G,K)$. $\qquad\square$

Corollary 10.2. *For every compact metric abelian group K and each strictly increasing sequence $S = \{a_n : n \in \mathbb{N}\}$ of integers, the set $\text{Weyl}(S,K)$ is an $F_{\sigma\delta}$-set in K.*

Proof. Follows from Proposition 10.1(ii) and the observation made after Definition 1.1. $\qquad\square$

The authors do not know of any example of a strictly increasing sequence $S = \{a_n : n \in \mathbb{N}\}$ of integers and a compact metric group K for which $\text{Weyl}(S,K)$ (and thus, $\mathbb{U}(S,K)$ as well) is not a G_δ-set. Of course, the case $K = \mathbb{T}$ is the most interesting here.

Item (iii) of our next remark shows that Proposition 10.1(i) does not hold for a non-metric group K.

Remark 10.3. (i) *Let E be a subset of an abelian group G, κ an uncountable cardinal and $n \in \mathbb{N} \setminus \{1\}$. Then the set $\Pi(E, \kappa, n) = \{\pi \in \text{Hom}(G, \mathbb{T}^\kappa) : \pi(E)$ is dense in $\mathbb{T}[n]^\kappa\}$ does not contain any non-empty G_δ-subset B of $\text{Hom}(G, \mathbb{T}^\kappa)$. Indeed, given such a B, there exist a countable subset Y of κ and $\pi = \{\pi_\alpha\}_{\alpha < \kappa} \in B$ such that $\pi_\alpha = 0$ for all $\alpha \in \kappa \setminus Y$. Let $\beta \in \kappa \setminus Y$ be arbitrary. Since $n \neq 1$, $\mathbb{T}[n] \neq \{0\}$. Since $\pi_\beta(E) = \{0\}$, the set $\pi_\beta(E)$ is not dense in $\mathbb{T}[n]$. Therefore, $\pi(E)$ cannot be dense in $\mathbb{T}[n]^\kappa$. Hence, $\pi \in B \setminus \Pi(E, \kappa, n)$.*

(ii) *The family $\Sigma_{G,\mathscr{E}}$ from Theorem 2.4 does not contain any non-empty G_δ-subset of $\widehat{G}^{\mathfrak{c}}$. Indeed, since $\mathscr{E} \neq \emptyset$, we can choose some $E \in \mathscr{E} \subseteq \mathfrak{T}(G)$. Then $\mathfrak{n}_E \neq 1$ by Remark 2.2(i). Finally, note that $\Sigma_{G,\mathscr{E}} \subseteq \Pi(E, \mathfrak{c}, \mathfrak{n}_E)$ and apply item (i).*

(iii) *If S is a subset of an abelian group G and X is an uncountable set, then $\mathbb{D}(S, \mathbb{T}^X)$ does not contain any non-empty G_δ-subset of $\widehat{G}^X$. Indeed, observe that $\mathbb{D}(S, \mathbb{T}^X) = \Pi(S, |X|, 0)$ and apply item (i).*

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